What Every Electrical Engineer Should Know about *Nonlinear* Circuits and Systems

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Outline

- Part 1: Motivation and Key Concepts (70)
- Part 2: Examples (5 x 18)
  - SPICE DC Analysis
  - Schmitt Trigger
  - Colpitts Oscillator
  - Injection-Locking
  - Digital Delta-Sigma Modulator

Part 1

Motivation and Key Concepts

Motivation and Key Concepts

- Motivation
  - Linear Systems
  - Nonlinear Systems
  - Nonlinear Effects
- Key Concepts in Nonlinear Dynamical Systems

Motivation

"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong."

Richard P. Feynman

Linear Systems

- Global behavior
  Unique solution; independent of initial conditions
- Frequencies don't mix
- Tractable
- Exact solution
- Quantitative analysis

Nonlinear Systems

- Local and global behavior
  - Coexisting solutions; dependence on initial conditions
  - Small- and large-signal behavior
- What does "frequency" mean?
- Normally intractable
- Approximate solutions
- *Qualitative* analysis

"Static" (Memoryless) Nonlinearities

- "Weak" and "strong" nonlinearities
  - Harmonic distortion
  - Saturation
  - Non-monotonicity
  - Hysteresis?

Nonlinear Effects

- Harmonic distortion (superharmonics)
- Saturation
- Amplitude limiting
- Bistability
- “Hysteresis”
- Subharmonics
- Synchronization

Motivation

“…even the theory of the simplest valve oscillator cannot *in principle* be reduced to the investigation of a linear differential equation and *requires the study of a nonlinear equation*."

…a linear equation, for example, cannot explain the fact that a valve oscillator, independently of the initial conditions, has a tendency to reach determined steady-state conditions.”

A.A. Andronov, A.A. Vitt, and S.E. Khaikin,
Theory of Oscillators, Pergamon Press, 1966

Examples

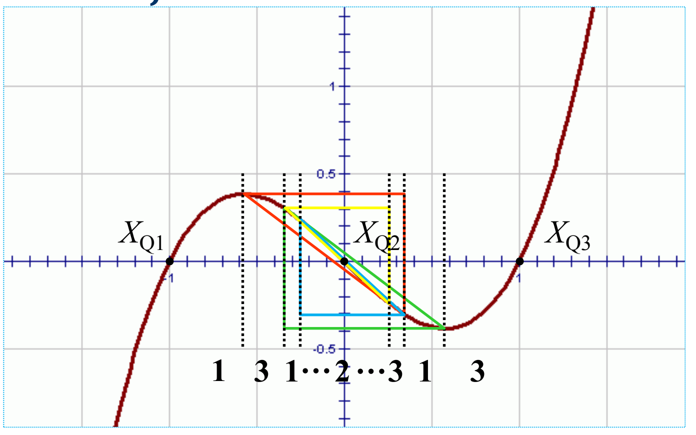
- SPICE DC Analysis
- Schmitt Trigger
- Colpitts Oscillator
- Injection-Locking
- Digital Delta-Sigma Modulator

Examples

- SPICE DC analysis: Why does my simulation not converge? Why does my oscillator not start in simulation?

Example: SPICE DC analysis

- The *basins of attraction* are intertwined…

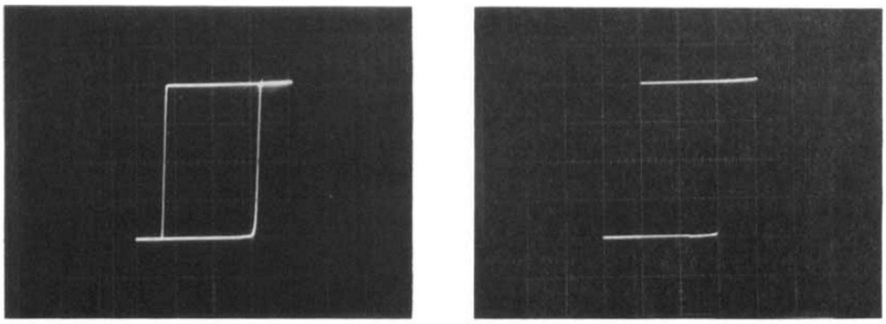


Examples

- Schmitt Trigger: What happens between the switching thresholds?

Example: Schmitt trigger

- Voltage transfer characteristic (measured)

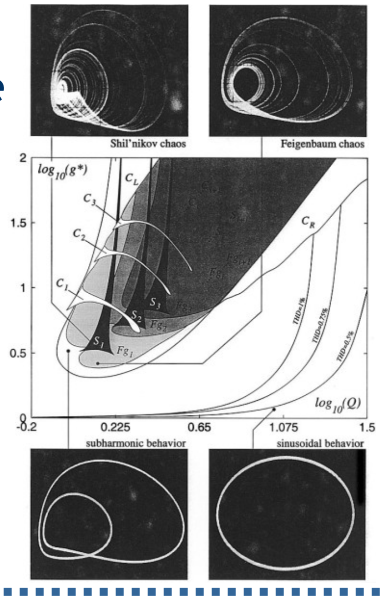


Examples

- Colpitts Oscillator: How large should the loop gain be?

Example: Colpitts Oscillator

- Design space



Examples

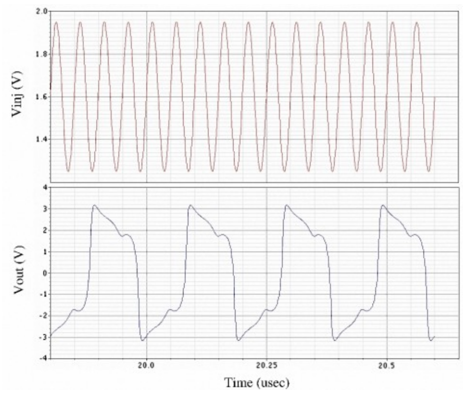
- Injection-Locking: What's the division ratio?

Example: Injection-Locking

- A point on line (d) of the sensitivity lines (Q=40).

f_{in}=*20* MHz

f_{out}=*5* MHz



$V_{in}\,@\,f_{in} \rightarrow$ Divide-by-4 ILFD $\rightarrow V_{out}\,@\,f_{out}$

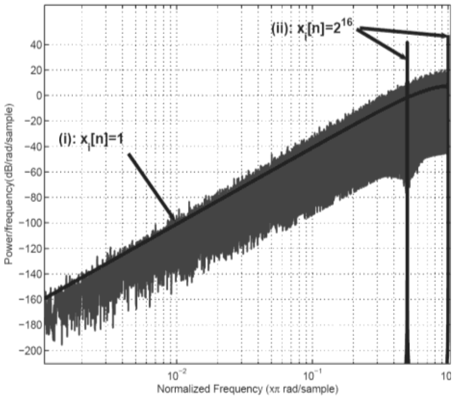
$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 10\,\text{MHz}$$

Examples

- Digital Delta Sigma Modulator: why do different values of the input yield different quantization noise spectra?

Example: DDSM

- Different *inputs* produce different spectra



MASH 1-1-1

wordlength: 17 bits

(i) $X=1$
(ii) $X=2^{16}$

Motivation

"I've wondered why it took us so long to catch on.
We saw it and yet we didn't see it.
Or rather *we were trained not to see it...*
The truth knocks on the door and you say, 'Go away, I'm looking for the truth,' and so it goes away…"

Robert M. Pirsig,
Zen and the Art of Motorcycle Maintenance, Corgi, 1974

Motivation

The “truths” of
Nonlinear Circuits and Systems
can provide some valuable insights…

Nonlinear Dynamical Systems

- Dynamical System
  - State
  - State equations
  - Trajectories
- Phase Portrait (effect of initial conditions)
  - Non-wandering sets
  - Attractors
  - Basins of attraction
- Bifurcation Diagram (parameters)
  - Local and global bifurcations

Nonlinear Dynamical Systems

- A dynamical system comprises a *state space* and a *dynamic*
- The *state space* is the set of possible states of the system; it is also called the "phase space"
- The *dynamic* describes the evolution of the state with time; it is also called the "state equations" or the "equations of motion"

State Equations (Dynamic)

- Continuous time

$$\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$$

- Discrete time

$$\mathbf{X}[n+1] = \mathbf{G}(\mathbf{X}[n]; \mu)$$

Nonlinear Dynamical Systems

- The continuous- (discrete-) time system is *nonlinear* if $\mathbf{F}$ ($\mathbf{G}$) is not linear in $\mathbf{X}$

Nonlinear Dynamical Systems

- A dynamical system follows a *trajectory* (orbit) through the state space
- The *flow* describes the evolution of all trajectories in the state space

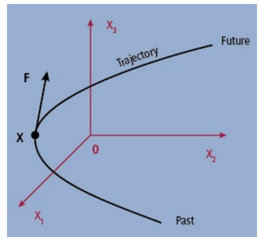
Trajectories (Orbits)

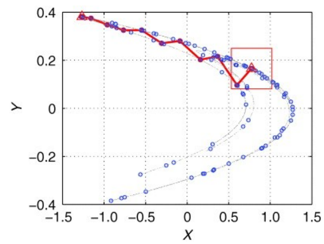
- Continuous time

$$\mathbf{X}(t) = \phi_t(\mathbf{X}(0); \mu)$$

- Discrete time

$$\mathbf{X}[n] = \phi_n(\mathbf{X}(0); \mu)$$





Nonlinear Dynamical Systems

- Nonlinear dynamics is concerned with:

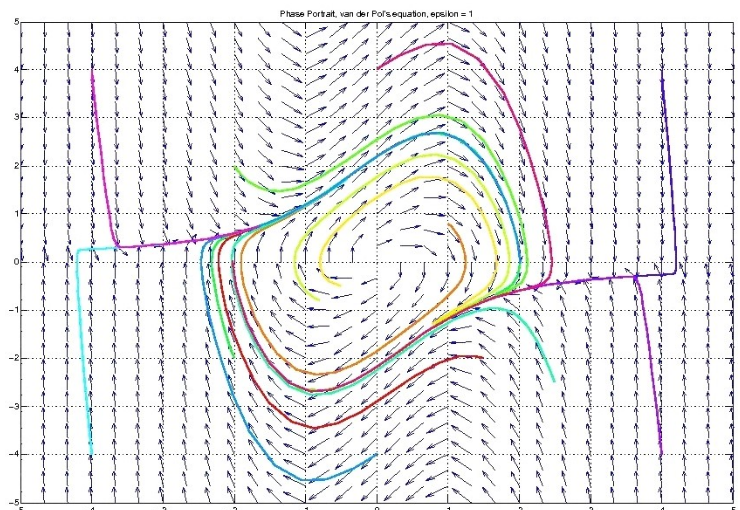
  (a) the *structure of the flow* (types of trajectories and their stability)

  (b) its *dependence on the parameters* of the system (structural stability and bifurcations)

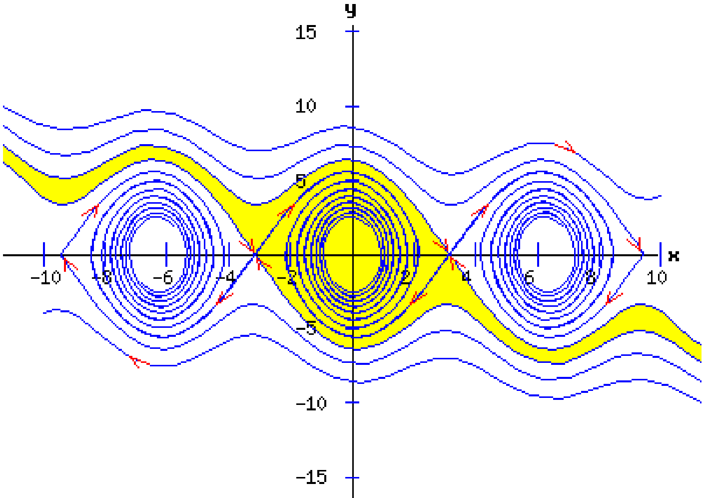
Phase Portrait and Bifurcation Diagram

- Phase portrait: Structure of the flow
- Bifurcation diagram: Dependence of the flow on the parameters

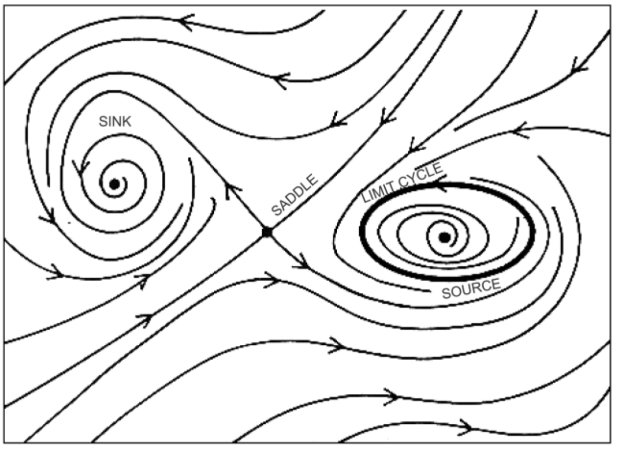
Phase Portrait



Phase Portrait

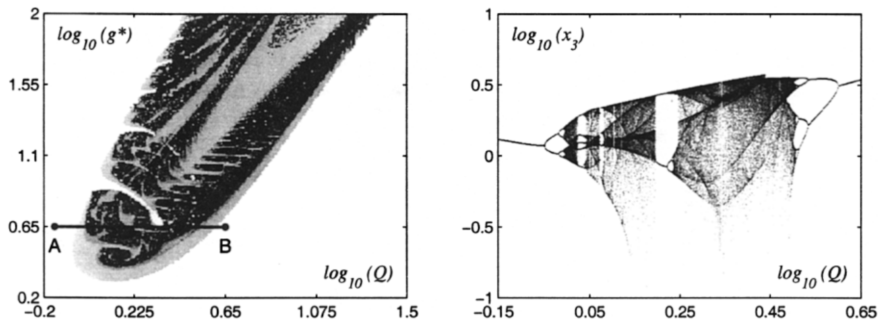


Phase Portrait

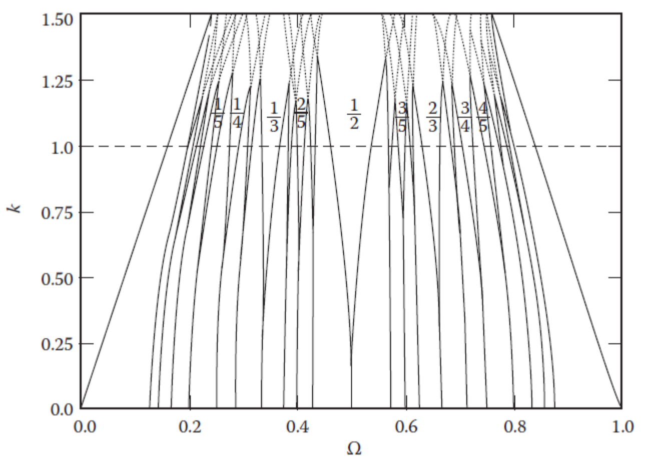


Source: Stewart (1989)

Bifurcation Diagram



Bifurcation Diagram



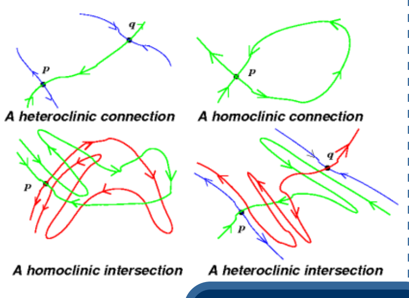
Structure of the Flow

- Types of trajectories
- Limit sets
- Stability of limit sets
- Attractors (“steady-state”)
- Basins of attraction/separatrices

Special Types of Trajectories

- Equilibrium (fixed) point
- Periodic trajectory (cycle)
- Quasiperiodic trajectory (torus)
- Chaotic trajectory

- Heteroclinic trajectory
- Homoclinic trajectory



Non-Wandering Sets

- Equilibrium (fixed) point
- Periodic trajectory (cycle)
- Quasiperiodic trajectory (torus)
- Chaotic trajectory

Equilibrium (Fixed) Point

- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

$$\mathbf{X}(t) = \mathbf{X}_Q$$

$$\mathbf{F}(\mathbf{X}_Q; \mu) = 0$$

- Discrete time $\mathbf{X}[n+1] = \mathbf{G}(\mathbf{X}[n]; \mu)$

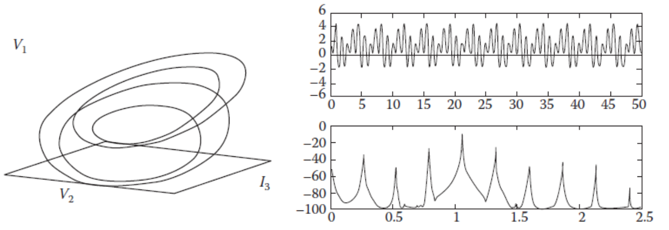
$$\mathbf{X}[n+1] = \mathbf{X}_Q$$

$$\mathbf{G}(\mathbf{X}_Q; \mu) = \mathbf{X}_Q$$

Periodic Trajectory (Cycle)

- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

$$\mathbf{X}(t+T) = \mathbf{X}(t)$$

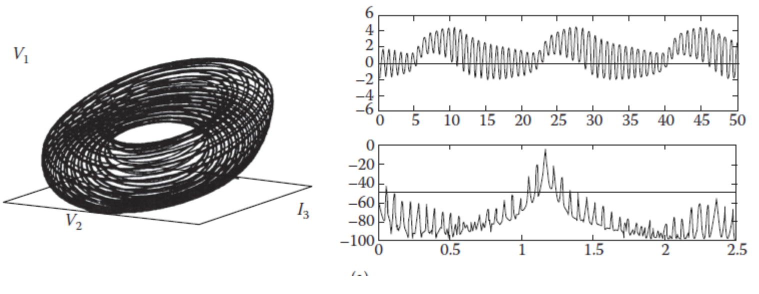


- Discrete time $\mathbf{X}[n+1] = \mathbf{G}(\mathbf{X}[n]; \mu)$

$$\mathbf{X}[n+N] = \mathbf{X}[n]$$

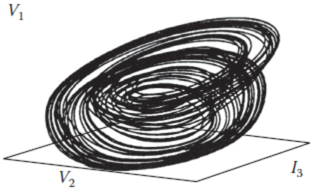
Quasiperiodic Trajectory (Torus)

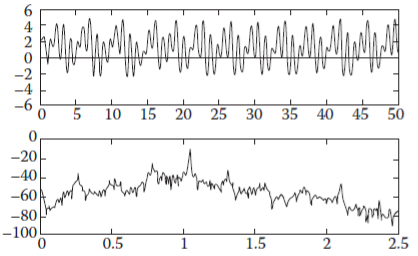
- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$



Chaotic Trajectory

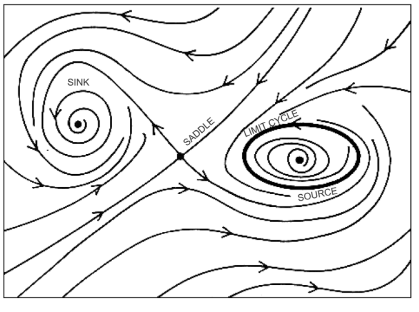
- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t);\mu)$





Stability of Non-Wandering Sets

- ω limit sets
- Attracting (or not)
- Basin of attraction



Source: Stewart (1989)

Stability of Non-Wandering Sets: Techniques

- Linearization (local behavior; “small-signal” analysis)
- Eigenvalues (negative or positive real parts [CT]; magnitudes less than or greater than unity [DT])

Stability of Non-Wandering Sets: Techniques

- Poincaré maps (reduce CT system to equivalent lower order DT system)
- Lyapunov exponents (generalized notion of stability)

Equilibrium (Fixed) Point

- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

$$\dot{\mathbf{x}}(t) = D_{\mathbf{X}}\mathbf{F}(\mathbf{X}_Q; \mu)\, \mathbf{x}(t)$$

- Discrete time $\mathbf{X}[n+1] = \mathbf{G}(\mathbf{X}[n]; \mu)$

$$\mathbf{x}[n+1] = D_{\mathbf{X}}\mathbf{G}(\mathbf{X}_Q; \mu)\, \mathbf{x}[n]$$

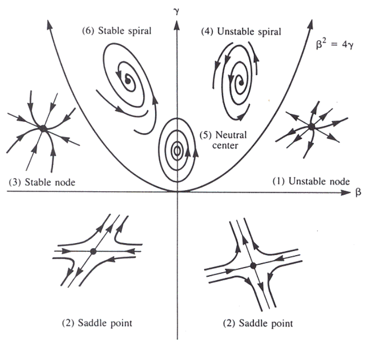
Linearization (Jacobian)

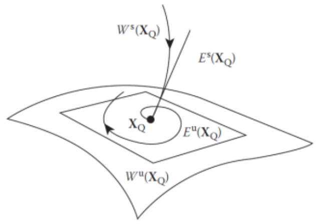
$$D_{\mathbf{X}}\mathbf{F} = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_2}{\partial X_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial X_1} & \frac{\partial F_N}{\partial X_2} & \cdots & \frac{\partial F_N}{\partial X_N} \end{pmatrix}$$

Equilibrium (Fixed) Point

- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

$$\dot{\mathbf{x}}(t) = D_{\mathbf{X}}\mathbf{F}(\mathbf{X}_Q; \mu)\, \mathbf{x}(t)$$

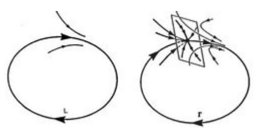




Periodic Trajectory (Cycle)

- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

  Poincaré map



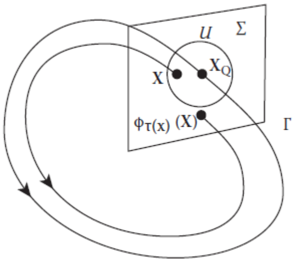
- Discrete time $\mathbf{X}[n+1] = \mathbf{G}(\mathbf{X}[n]; \mu)$

$$\mathbf{x}[n+1] = D_{\mathbf{X}}\mathbf{G}^N(\mathbf{X}_Q; \mu)\, \mathbf{x}[n]$$

Poincaré Map

- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

  Poincaré map

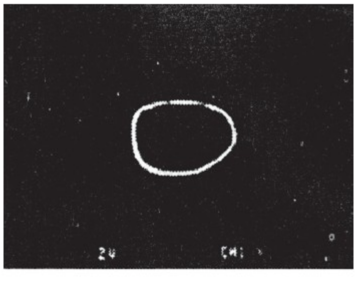


$$\mathbf{X}[n+1] = \mathbf{G}(\mathbf{X}[n]; \mu)$$

Quasiperiodic Trajectory (Torus)

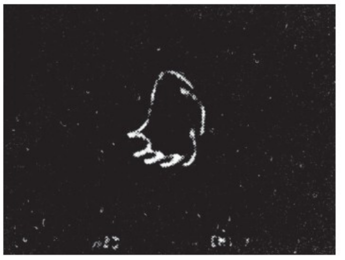
- Continuous time $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t); \mu)$

  Poincaré map

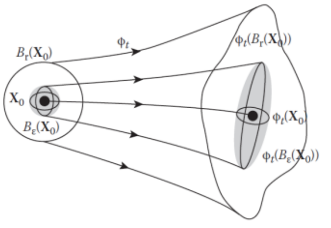


Chaotic Trajectory

- Poincaré map



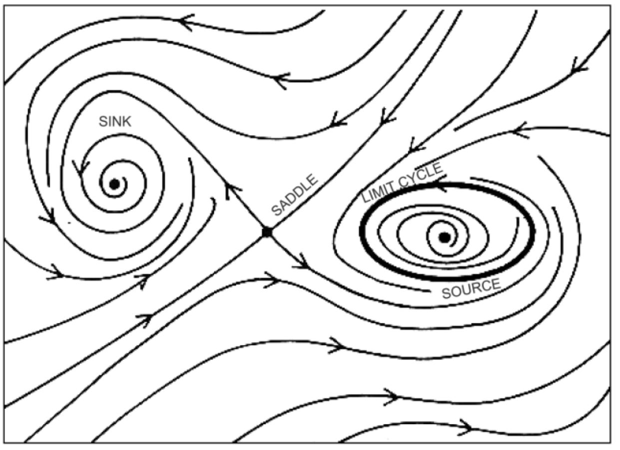
- Lyapunov exponents



Summary: Phase Portrait

- Flow
- Non-wandering sets
- Special trajectories (heteroclinic and homoclinic connections)
- Separatrices
- ω limit sets (attractors; "steady states")
- Basins of attraction

Phase Portrait



Source: Stewart (1989)

Structural Stability

- A *structurally stable* dynamical system is one for which a small change in a parameter does not change the dynamics qualitatively
- A *bifurcation* occurs if the qualitative dynamics change; the system is not structurally stable at the bifurcation point

Bifurcations

- Local bifurcation: The stability of an equilibrium (or fixed) point changes
- Global bifurcation: A 'larger' invariant set, such as a periodic orbit, collides with an equilibrium

Bifurcations

- Local bifurcations
  - Saddle-node (fold, tangent) [Schmitt trigger]
  - Transcritical
  - Pitchfork [Bistable]
  - Period-doubling (flip)
  - Hopf [“Sinusoidal” oscillator]
- Global bifurcations
  - Homoclinic: limit cycle collides with a saddle
  - Heteroclinic: limit cycle collides with two or more saddles
  - Blue sky catastrophe: limit cycle collides with an unstable cycle

Local Bifurcations

- Saddle-node (fold, tangent)

  A stable and an unstable equilibrium point collide and disappear

- Pitchfork

  A stable equilibrium point goes unstable and two new stable equilibria appear

Local Bifurcations

- Period-doubling (flip)

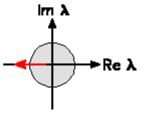
  A stable fixed point goes unstable with an eigenvalue equal to -1

- Hopf

  A stable equilibrium point becomes unstable and a stable cycle appears

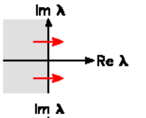
period doubling bifurcation

discrete dynamical system

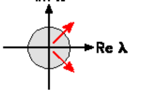


Hopf bifurcation

continuous dynamical system



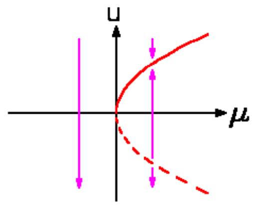
discrete dynamical system



Saddle-Node Bifurcation [CT]

- State equation $\dot{X}(t) = \mu - X^2(t)$
- Equilibrium points $X_Q = \pm\sqrt{\mu}$
- Stability $\dot{x}(t) = (-2X_Q)\, x(t)$

saddle-node bifurcation



$$\dot{X}(t) = \mu - X^2(t)$$

Saddle-Node Bifurcation [DT]

- State equation $X[n+1] = X[n] + \mu - X^2[n]$
- Equilibrium points $X_Q = \pm\sqrt{\mu}$
- Stability $x[n+1] = (1 - 2X_Q)\, x[n]$

Pitchfork Bifurcation [CT]

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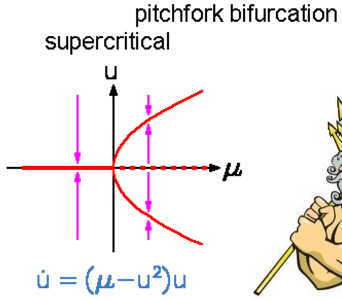
- State equation $\dot{X}(t) = \mu X(t) - X^3(t)$
- Equilibrium points $X_Q = 0, \pm\sqrt{\mu}$
- Stability $\dot{x}(t) = (\mu - 3{X_Q}^2)\, x(t)$



www.tyndall.ie

Pitchfork Bifurcation [DT]

64/168

- State equation $X[n+1] = X[n] + \mu X[n] - X^3[n]$
- Equilibrium points $X_Q = 0, \pm\sqrt{\mu}$
- Stability $x[n+1] = (1 + \mu - 3{X_Q}^2)\, x[n]$

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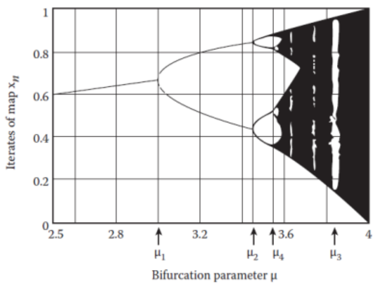
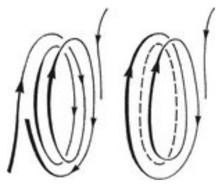
Period-Doubling Bifurcation [DT]

- State equation $X[n+1] = \mu - X[n] - X^2[n]$
- Equilibrium point(s) $X_Q = -1 \pm \sqrt{1+\mu}$
- Stability $x[n+1] = (-1 - 2X_Q)\,x[n]$





Hopf Bifurcation [CT]

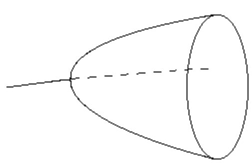
- State equation

$$
\begin{aligned}
\dot{X}_1(t) &= \mu X_1(t) - \omega X_2(t) + (-X_1(t) - \beta X_2(t))({X_1}^2(t) + {X_2}^2(t)) \\
\dot{X}_2(t) &= \omega X_1(t) + \mu X_2(t) + (\beta X_1(t) - X_2(t))({X_1}^2(t) + {X_2}^2(t))
\end{aligned}
$$

- Equilibrium point

$$
\begin{aligned}
X_{1Q} &= 0 \\
X_{2Q} &= 0
\end{aligned}
$$

- Stability



Local Bifurcations

- Saddle-node (fold, tangent)
  A stable and an unstable equilibrium point collide and disappear
- Pitchfork
  A stable equilibrium point goes unstable and two new stable equilibria appear
- Period-doubling (flip)
  A stable fixed point goes unstable with eigenvalue equal to -1
- Hopf
  A stable equilibrium point becomes unstable and a stable cycle appears

Summary

- Part 1: Motivation and Basic Concepts
  Coexisting solutions
  Bistability
  "Hysteresis"
  Synchronization

  State space trajectories
  Limit sets (steady-state solutions) and basins of attraction

  Structural stability
  Bifurcations (saddle-node, pitchfork, period-doubling, Hopf)

References: Nonlinear Dynamics

[K93a] M.P. Kennedy. Three steps to chaos part I: Evolution. *IEEE Trans. Circuits and Systems-Part I: Fundamental Theory and Applications*, 40(10):640-656, October 1993.

[K93b] M.P. Kennedy. Three steps to chaos part II: A Chua's circuit primer. *IEEE Trans. Circuits and Systems-Part I: Fundamental Theory and Applications*, 40(10):657-674, October 1993.

[K94] M.P. Kennedy. Basic concepts of nonlinear dynamics and chaos. In C. Toumazou, editor, *Circuits and Systems Tutorials*, pages 289-313. IEEE Press, London, 1994.

References: Nonlinear Dynamics

[NBb] A.H. Nayfeh and B. Balachandran. Applied Nonlinear Dynamics. Wiley, 1995.

[S01] S.H. Strogatz. Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering. Perseus Books, 2001.

Part 2: Examples

- SPICE DC Analysis (intertwined basins of attraction)
- Schmitt Trigger (Saddle-node bifurcation)
- Colpitts Oscillator (Hopf bifurcation)
- Injection-Locking (Arnold tongues)
- Digital Delta-Sigma Modulator (Coexisting solutions)

Acknowledgements

- FIRB Italian National Research Program RBAP06L4S5
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Fondúireacht Eolaíochta Éireann
Science Foundation Ireland

Part 2

Examples

Examples

- SPICE DC Analysis (intertwined basins of attraction)
- Schmitt Trigger (Saddle-node bifurcation)
- Colpitts Oscillator (Hopf bifurcation)
- Injection-Locking (Arnold tongues)
- Digital Delta-Sigma Modulator (Coexisting solutions)

Example: SPICE DC analysis

- Local: Equilibrium ("operating") point
- Global: Coexisting stable equilibrium points; multiple basins of attraction; complex separatrices

Example: SPICE DC analysis

- At each time point, SPICE solves a nonlinear circuit using Modified Nodal Analysis (MNA) and the Newton-Raphson algorithm

```
Transient loop :
while the simulation is not over
  Formulate companion models for energy storage
     components, using current operating point
  Newton loop :
  while the convergence is not achieved
    Formulate companion models for non-linear components,
        using current operating point
    Solve for new operating point
  end while
end while
```

Example: SPICE DC analysis

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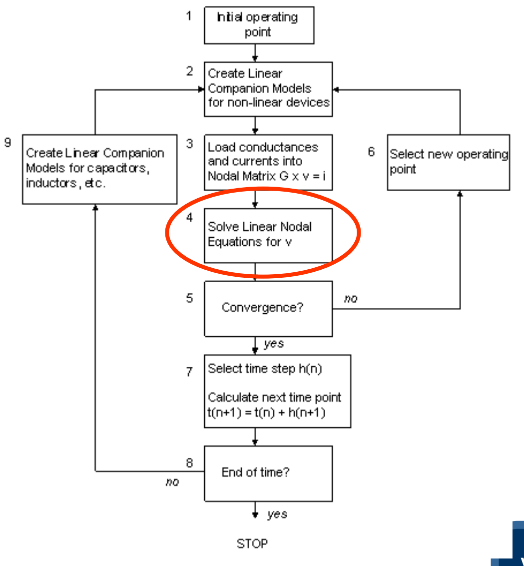

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Example: SPICE DC analysis

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- Modified Nodal Analysis (MNA)

$$\mathbf{M}(\mathbf{X}) = \mathbf{S}$$

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Example: SPICE DC analysis

- Use Newton-Raphson to solve

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}$$

where

$$\mathbf{F}(\mathbf{X}) = \mathbf{M}(\mathbf{X}) - \mathbf{S}$$

Example: SPICE DC analysis

- Newton-Raphson is a *dynamical system*

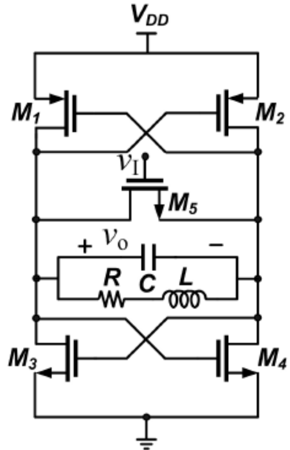
$$\mathbf{X}[n+1] = \mathbf{X}[n] - D_{\mathbf{X}}\mathbf{F}^{-1}(\mathbf{X}[n])\mathbf{F}(\mathbf{X}[n])$$

with equilibrium (fixed) point(s) at

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}$$

Example: SPICE DC analysis

- CMOS ILFD with direct injection

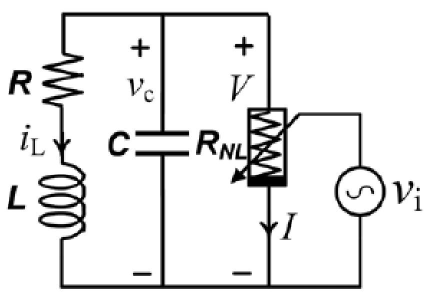


Example: SPICE DC analysis

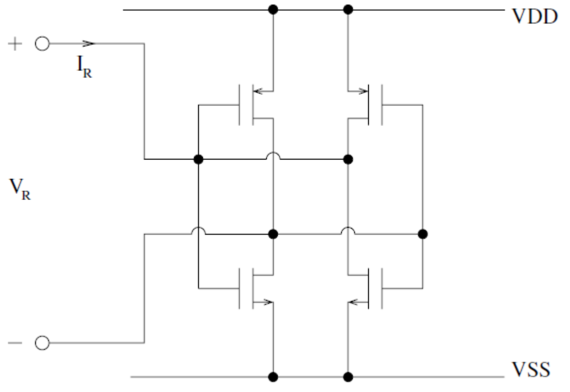
- Simplified model

$$C\frac{dv_C}{dt} = -i_L - f\left(v_C, v_I\right),$$

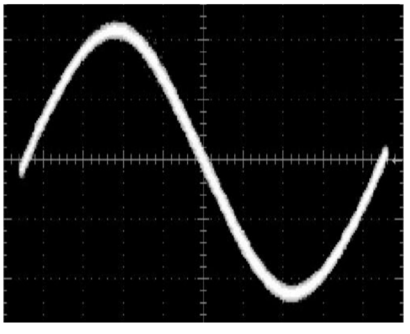
$$L\frac{di_L}{dt} = v_C - R\,i_L,$$

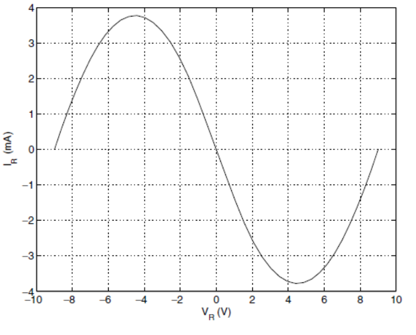


- Nonlinear resistor



- Nonlinear resistor





$$f(V) = G_a V \left(1 - \frac{V^2}{V_{\max}^2}\right)$$

Example: SPICE DC analysis

- Modified Nodal Analysis (MNA)

$$\frac{1}{R}V + f(V) = 0$$

where

$$f(V) = G_a V \left(1 - \frac{V^2}{V_{\max}^2}\right)$$

Example: SPICE DC analysis

- Use Newton-Raphson to solve

$$F(V) = 0$$

where

$$F(V) = \left(\frac{1}{R} + G_a\right)V + \left(-\frac{G_a}{V_{\max}^2}\right)V^3$$

Example: SPICE DC analysis

- Newton-Raphson is a dynamical system

$$X[n+1] = X[n] - \frac{F(X[n])}{\frac{d}{dX}F(X[n])}$$

with

$$F(X) = c_1 X + c_3 X^3$$

and

$$c_1 = \left(\frac{1}{R} + G_a\right), \; c_3 = \left(-\frac{G_a}{V_{\max}^2}\right)$$

Example: SPICE DC analysis

- State equation

$$X[n+1] = X[n] - \left(\frac{c_1 X[n] + c_3 X^3[n]}{c_1 + 3c_3 X^2[n]}\right)$$

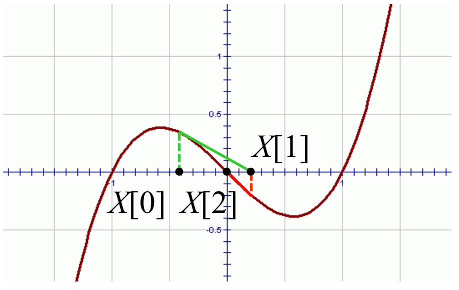
- Equilibrium (fixed) points

$$X_Q = 0, \pm V_{\max}\sqrt{1 + \frac{1}{G_a R}}$$

Example: SPICE DC analysis

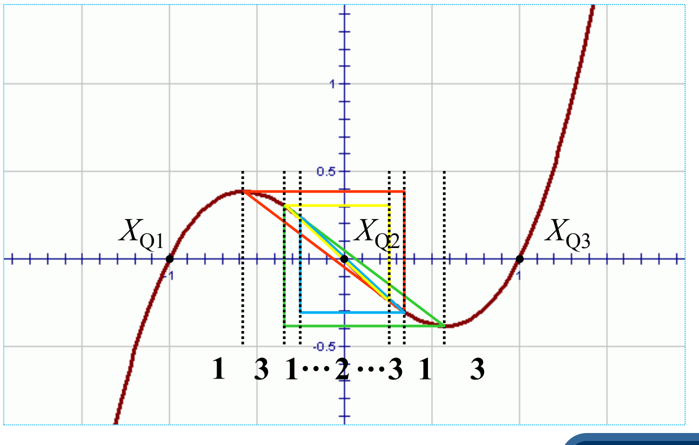
- The next state is obtained from the current state by following the tangent…

$$X[n+1] = X[n] - \frac{F(X[n])}{\frac{d}{dX}F(X[n])}$$



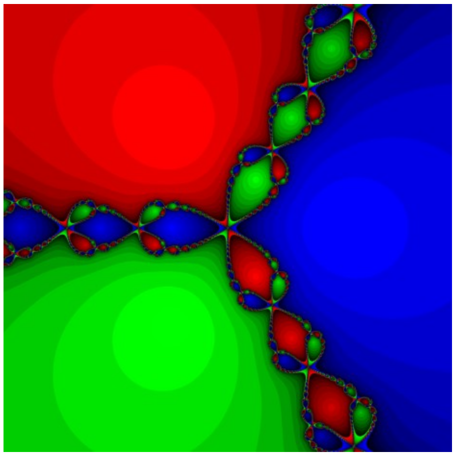
Example: SPICE DC analysis

- The basins of attraction are intertwined…



Example: SPICE DC analysis

- Intertwining can be very complicated…



Example: SPICE DC analysis

- Issues
  - Oscillation (of algorithm)
  - Unboundedness
  - “Missed” solution
  - No oscillation (of circuit)

Example: SPICE DC analysis

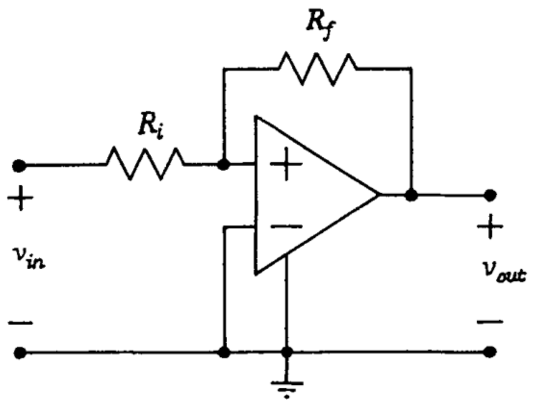
- Local: Equilibrium ("operating") point
- Global: Coexisting stable equilibrium points; multiple basins of attraction; complex separatrices

Example: Schmitt trigger

- Local: Stable equilibrium point
- Global: Coexisting stable and unstable equilibrium points ("bistable")
- Structural: Saddle-node (fold) bifurcations at transition points

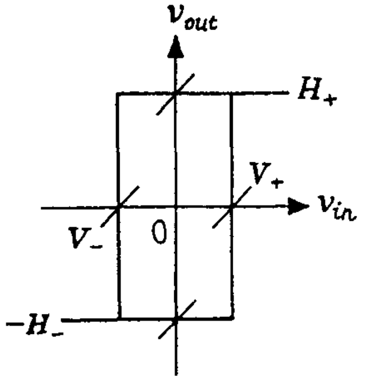
Example: Schmitt trigger

- Circuit diagram

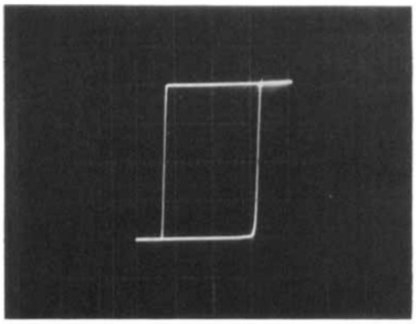


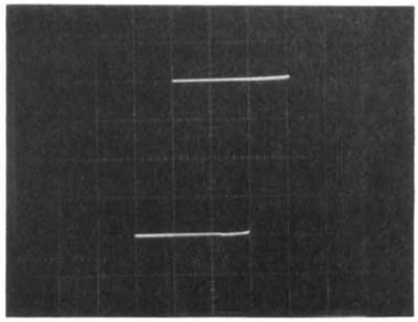
Example: Schmitt trigger

- Voltage transfer characteristic

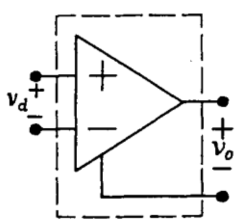


- Voltage transfer characteristic (measured)

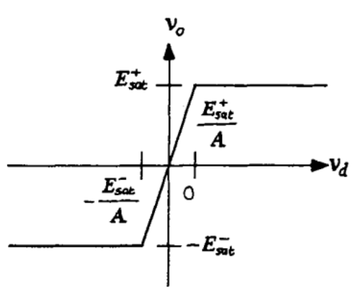




- Simplified model



(a)

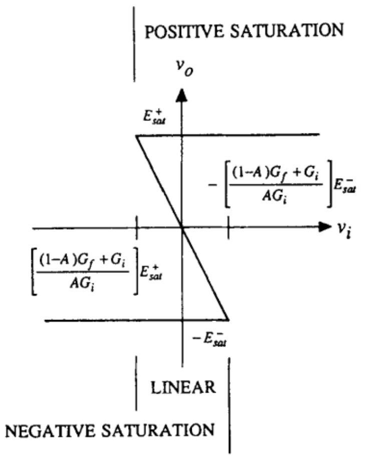


(b)

Figure 10. (a) Finite gain op amp model[12] and (b) its three-segment PWL voltage transfer characteristic

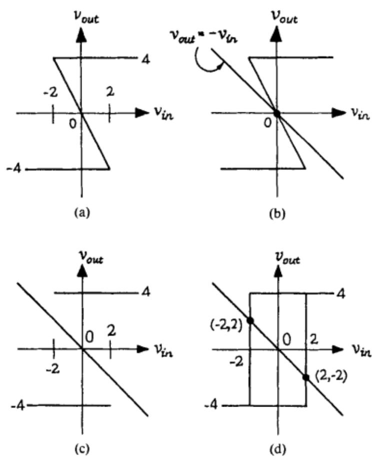
Example: Schmitt trigger

- Simplified model



Example: Schmitt trigger

- Experimental verification



Example: Schmitt trigger

- Measurement circuit

$$u = -(v_{\text{out}} + v_{\text{in}})$$

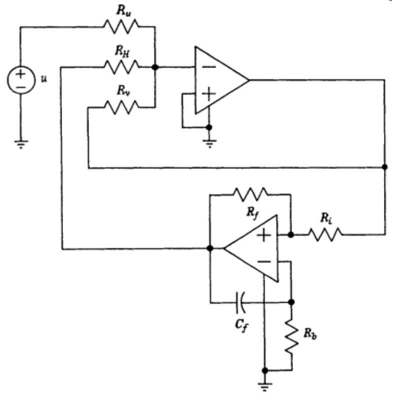
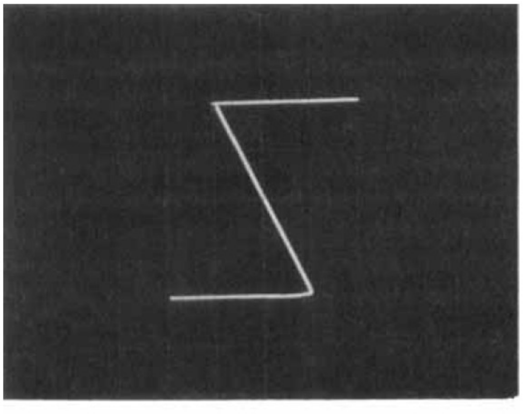


Figure 14. Op amp network for determining the TC of the Schmitt trigger of Figure 6. The circuit consists of an independent voltage source u (top left), an inverting summer (top) and a Schmitt trigger with AC feedback (bottom). $R_u = 20\ \text{k}\Omega$, $R_H = 20\ \text{k}\Omega$, $R_v = 20\ \text{k}\Omega$, $R_f = 20\ \text{k}\Omega$, $R_i = 10\ \text{k}\Omega$, $R_b = 470\ \Omega$, $C_f = 470\ \text{pF}$, op amps are 1/4 TL074

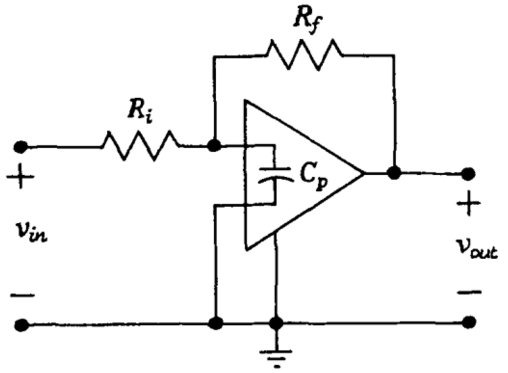
Example: Schmitt trigger

- Voltage transfer characteristic (measured)



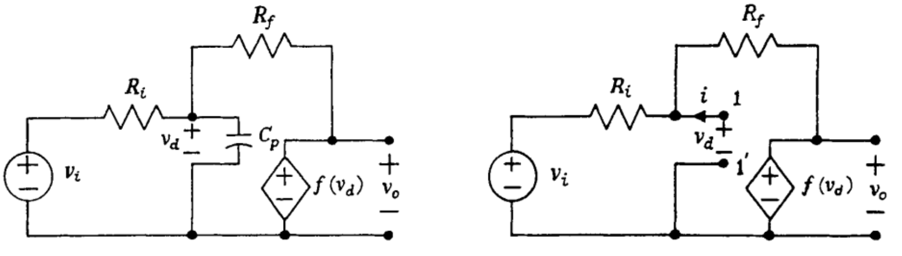
Example: Schmitt trigger

- One parasitic capacitor is sufficient to explain the switching mechanism…



Example: Schmitt trigger

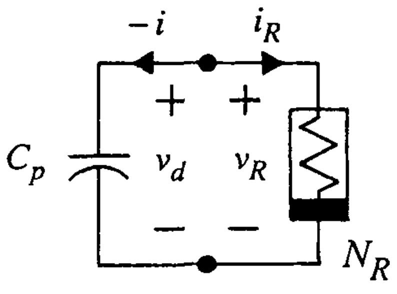
- Equivalent circuit



Example: Schmitt trigger

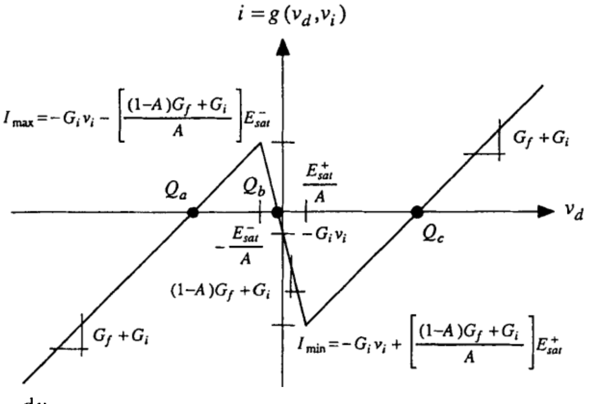
- Equivalent circuit

$$C_p \frac{dv_d}{dt} = G_i v_i + [G_f f(v_d) - G_f v_d - G_i v_d] = -g(v_d, v_i)$$



Example: Schmitt trigger

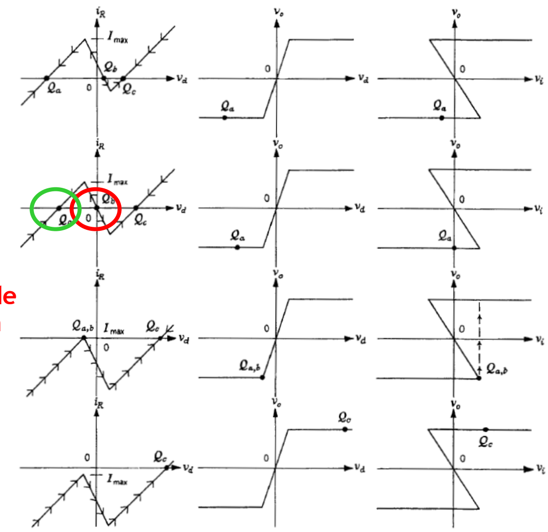
- Nonlinear resistor seen by parasitic



$$C_p \frac{dv_d}{dt} = G_i v_i + [G_f f(v_d) - G_f v_d - G_i v_d] = -g(v_d, v_i)$$

Example: Schmitt trigger

Saddle-node bifurcation



Example: Schmitt trigger

- Simulation issues
  - Equilibrium point
  - Coexisting solutions

Example: Schmitt trigger

- Local: Stable equilibrium point
- Global: Coexisting stable and unstable equilibrium points (“bistable”)
- Structural: Saddle-node (fold) bifurcations at transition points

References: Hysteresis

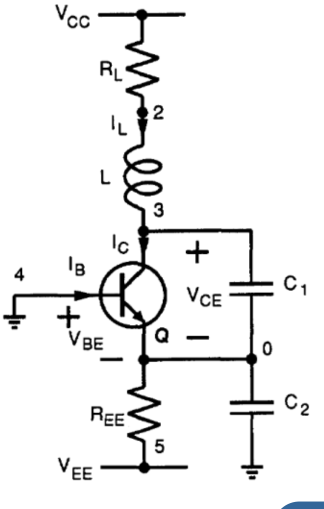
[KC91] M.P. Kennedy and L.O. Chua. Hysteresis in electronic circuits: A circuit theorist's perspective. *Int. J. Circuit Theory Appl.*, 19(5):471-515, 1991.

Example: Colpitts Oscillator

- Local: Unstable equilibrium point; stable limit cycle
- Global: Amplitude-limiting mechanism
- Structural: Hopf bifurcation; period-doubling (flip) bifurcation

Example: Colpitts Oscillator

- Circuit diagram



Example: Colpitts Oscillator

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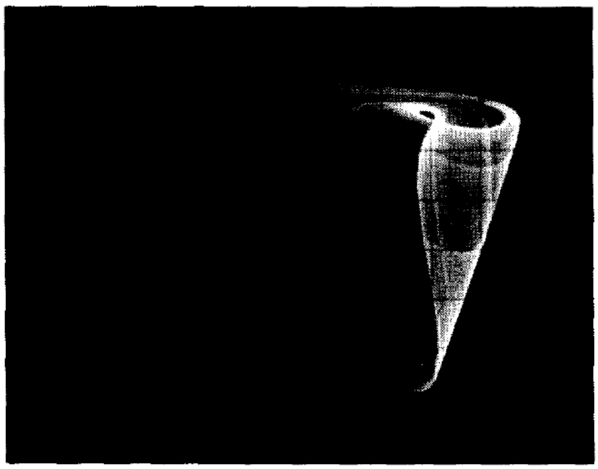

Fig. 2. Suspected chaotic attractor in the BJT Colpitts oscillator of Fig. 1. Horizontal axis: V_{CE} (V); vertical axis V_{BE} (V).

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Example: Colpitts Oscillator

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- SPICE deck

```
vcc 1 4 pwl(0 0 1n 5 5m 5)
vee 5 4 dc -5
q1  3 4 0 Q2N2222A
rl  1 2 35
l1  2 3 98.5u
c1  3 0 54n
c2  4 0 54n
ree 0 5 400
```

```
.model Q2N2222A NPN(Is=14.34f
+ Xti=3 + Eg=1.11 Vaf=74.03
+ Bf=255.9 Ne=1.307 Ise=14.34f
+ Ikf=.2847 Xtb=1.5 Br=6.092
+ Nc=2 Isc=0 Ikr=0 Rc=1
+ Cjc=7.306p Mjc=.3416 Vjc=.75
+ Fc=.5 Cje=22.01p Mje=.377
+ Vje=.75 Tr=46.91n Tf=411.1p
+ Itf=.6 Vtf=1.7 Xtf=3 Rb=10)

.options reltol=1e-4
.tran 10n 4m 3m
.end
```

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Example: Colpitts Oscillator

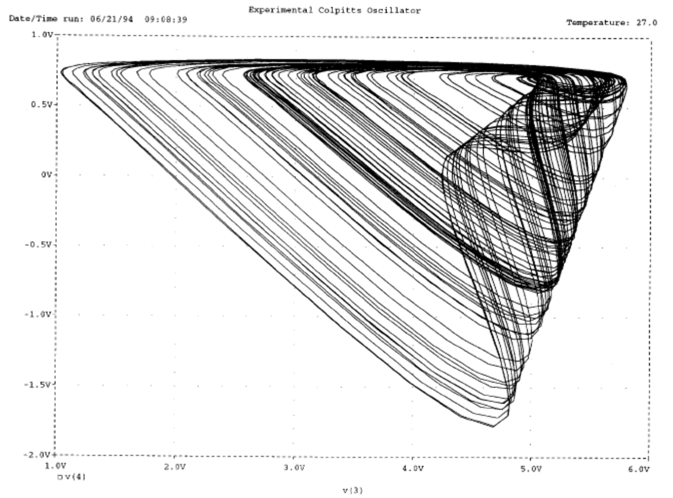
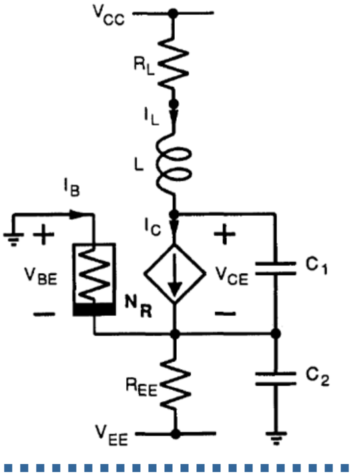


Fig. 4. PSpice simulation of the BJT Colpitts oscillator using the deck shown in Fig. 3. Horizontal axis: V(3) = V_{CE} (V); vertical axis: V(4) = V_{BE} (V). Note that V_{CE} is always positive.

Example: Colpitts Oscillator

- Simplified circuit diagram



- State equations

$$
\begin{aligned}
C_1 \frac{V_{CE}}{dt} &= I_L - I_C \\
C_2 \frac{V_{BE}}{dt} &= -\frac{V_{EE} + V_{BE}}{R_{EE}} - I_L - I_B \\
L \frac{I_L}{dt} &= V_{CC} - V_{CE} + V_{BE} - I_L R_L.
\end{aligned}
$$

$$
\begin{aligned}
I_B &= \begin{cases} 0 & \text{if} \quad V_{BE} \leq V_{TH} \\ \frac{V_{BE} - V_{TH}}{R_{ON}} & \text{if} \quad V_{BE} > V_{TH}, \end{cases} \\
I_C &= \beta_F I_B
\end{aligned}
$$

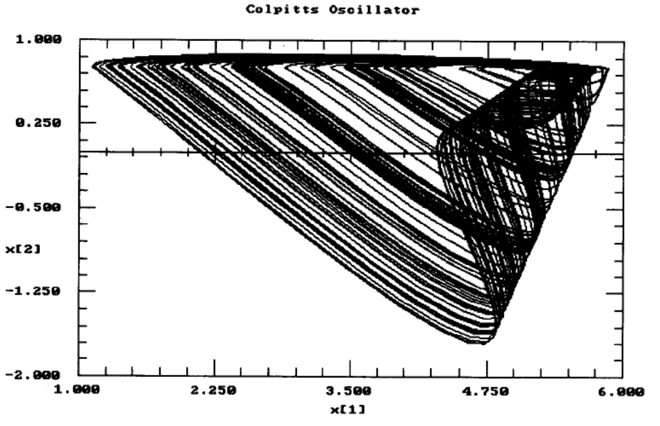
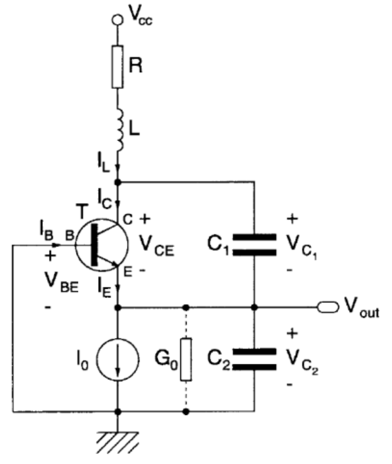


Fig. 5. INSITE simulation of the piecewise-linear BJT Colpitts oscillator model. Model parameters are: V_{CC} = 5 V, R_L = 35 Ω, L = 98.5 μH, C_1 = 54 nF, C_2=54 nF, R_{EE} = 400 Ω, and V_{EE} = -5 V. The BJT is specified by three parameters: V_{TH} = 0.75 V, R_{ON} = 100Ω, and β_F = 200. Horizontal axis: x[1] = V_{CE} (V); vertical axis: x[2] = V_{BE} (V).

Example: Colpitts Oscillator

- Alternative topology



Example: Colpitts Oscillator

- State equations

$$C_1\frac{dV_{C_1}}{dt} = -f(V_{C_2}) + I_L$$
$$C_2\frac{dV_{C_2}}{dt} = I_L - I_0$$
$$L\frac{dI_L}{dt} = -V_{C_1} - V_{C_2} - RI_L + V_{CC}$$

$$f(V_{C_2}) = I_S \exp\left(-\frac{V_{C_2}}{V_T}\right)$$

Example: Colpitts Oscillator

- Normalized state equations

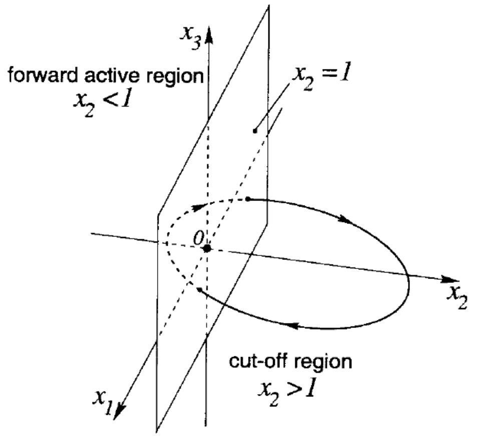
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \dfrac{g^*}{Q(1-k)}[-n(x_2)+x_3] \\ \dfrac{g^*}{Qk}x_3 \\ -\dfrac{Qk(1-k)}{g^*}[x_1+x_2] - \dfrac{1}{Q}x_3 \end{pmatrix}$$

$$n(x_2) = \exp(-x_2) - 1$$

$$k = C_2/(C_1 + C_2)$$

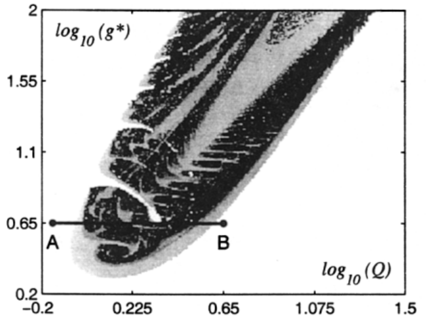
Example: Colpitts Oscillator

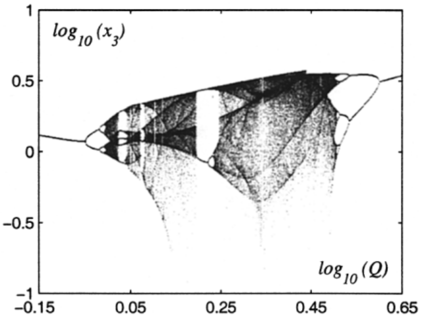
- Phase portrait



Example: Colpitts Oscillator

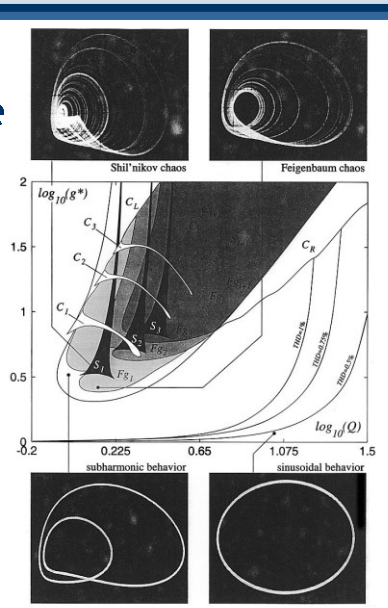
- Bifurcation diagram





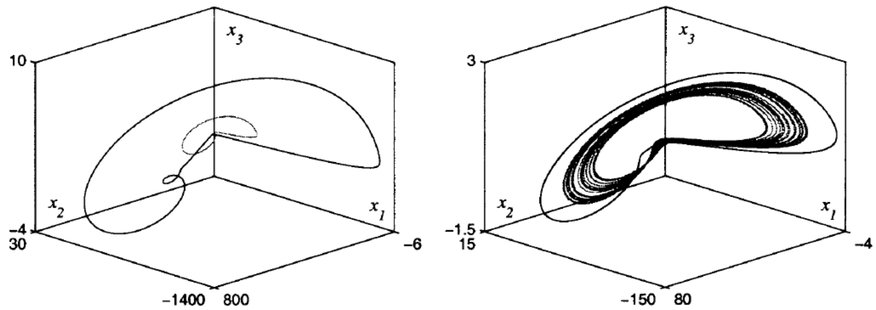
Example: Colpitts Oscillator

- Design space



Example: Colpitts Oscillator

- Simulation issues
  - Equilibrium point
  - Coexisting solutions



Example: Colpitts Oscillator

- Local: Unstable equilibrium point; stable limit cycle
- Global: Amplitude-limiting mechanism
- Structural: Hopf bifurcation; period-doubling (flip) bifurcation

References: Colpitts Oscillator

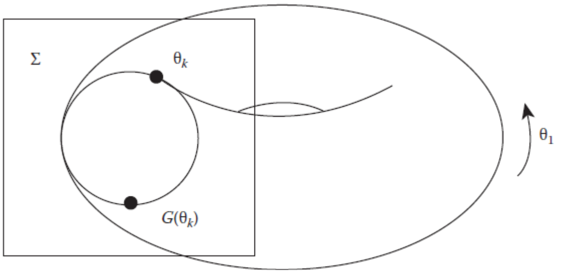
[K94] M.P. Kennedy. Chaos in the Colpitts oscillator. *IEEE Trans. Circuits and Systems-Part I*, 41(11):771-774, November 1994.

[MdeFK99] G.M. Maggio, O.de Feo, and M.P. Kennedy. Nonlinear analysis of the Colpitts oscillator with applications to design. *IEEE Trans. Circuits and Systems-Part I: Fundamental Theory and Applications*, CAS-46(9):1118-1130, September 1999.

Example: Injection-Locking

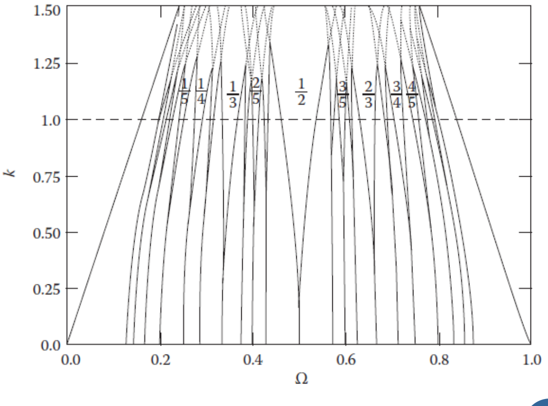
- Local: Stable limit cycle
- Global: Arnold tongues
- Structural: Fold (?) bifurcation

- Model of injection-locking: The Circle map



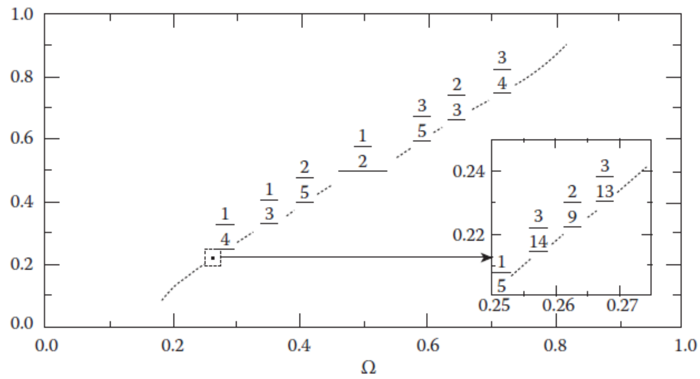
$$\theta_{k+1} = \left(\theta_k + \frac{K}{2\pi}\sin(2\pi\theta_k) + \Omega\right) \bmod 1, \quad k = 0, 1, 2, \ldots$$

- Bifurcation diagram (Arnold tongues)



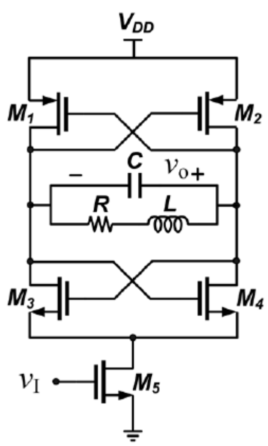
Example: Injection-Locking

- Devil's staircase



Example: Injection-Locking

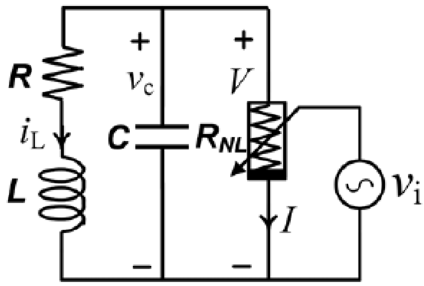
- CMOS ILFD with tail coupling



Example: Injection-Locking

- Simplified model

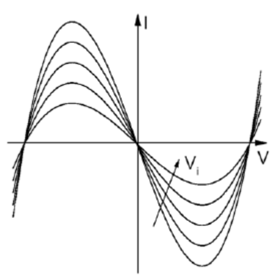
$$C\frac{dv_C}{dt} = -i_L - f\left(v_C, v_I\right),$$
$$L\frac{di_L}{dt} = v_C - R\,i_L,$$



Example: Injection-Locking

- Nonlinearity

$$I_N = f\left(V_N, v_I\right) = -a\left(v_I\right)\cdot V_N \cdot \left(1 - \left(\frac{V_N}{V_{DD}}\right)^2\right)$$



$$\begin{aligned} a\left(v_I\right) &\approx a\left(V_{dc}\right) + \left.\frac{\partial a\left(v_I\right)}{\partial v_I}\right|_{v_I=V_{dc}} \cdot V_i \cdot g\left(\omega_i t\right) \\ &= a\left(V_{dc}\right)\left(1 + m_f \cdot g\left(\omega_i t\right)\right), \end{aligned}$$

Example: Injection-Locking

- Normalized model $x = \frac{v_C}{V_{DD}}, \quad y = \frac{R\, i_L}{V_{DD}}, \quad \tau = \frac{t}{\sqrt{LC}},$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}, \quad G = R \cdot a(V_{dc}), \quad \omega = \omega_i \sqrt{LC},$$

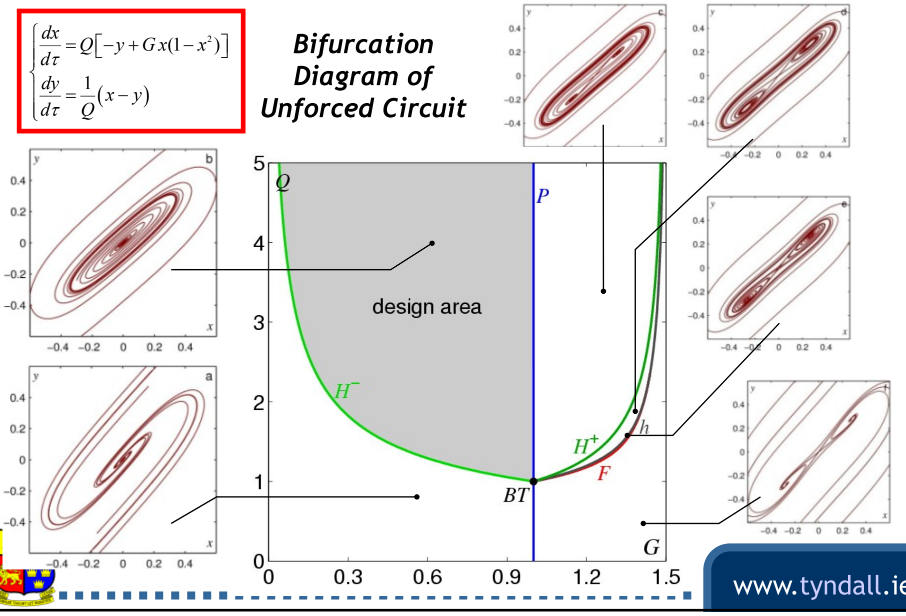
$$\frac{dx}{d\tau} = Q\left[-y + \mathcal{G}(\tau)\, x\left(1 - x^2\right)\right],$$
$$\frac{dy}{d\tau} = \frac{1}{Q}(x - y),$$

$$\mathcal{G}(\tau) = G\left(1 + m_f \cdot g(\omega\tau)\right)$$

Example: Injection-Locking

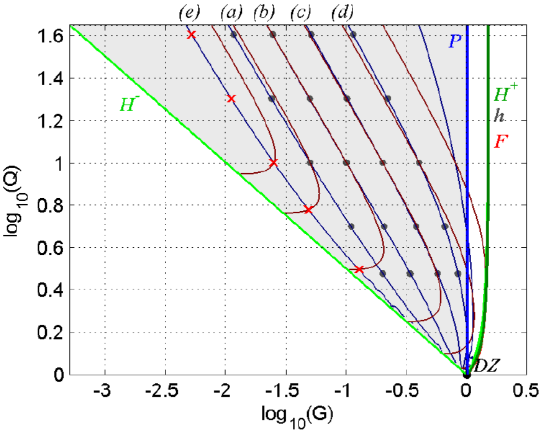
$$\begin{cases} \frac{dx}{d\tau} = Q\left[-y + Gx(1-x^2)\right] \\ \frac{dy}{d\tau} = \frac{1}{Q}(x-y) \end{cases}$$

Bifurcation Diagram of Unforced Circuit



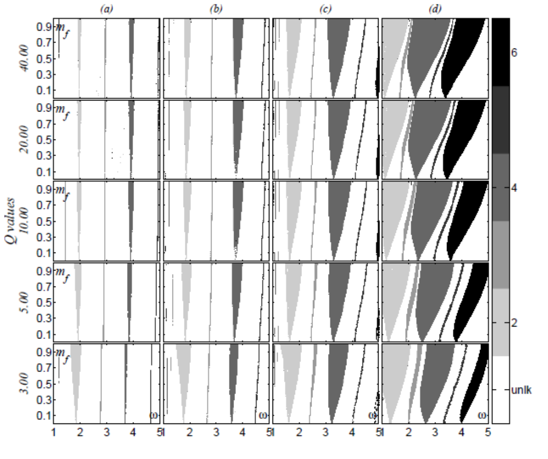
Example: Injection-Locking

- Bifurcation diagram (simulated)



Example: Injection-Locking

- Bifurcation diagram (simulated)



Example: Injection-Locking

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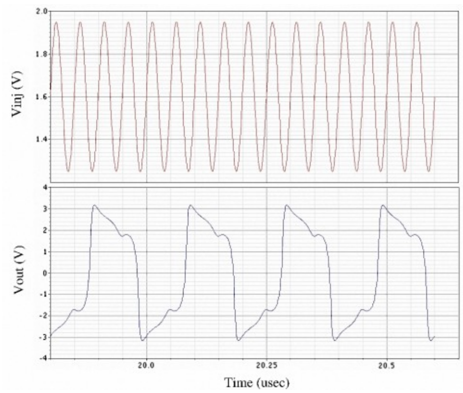
- A point on line (d) of the sensitivity lines (Q=40).

f_{in}=20 MHz

f_{out}=5 MHz



$V_{in}\,@\,f_{in} \rightarrow$ Divide-by-4 ILFD $\rightarrow V_{out}\,@\,f_{out}$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 10\,\text{MHz}$$

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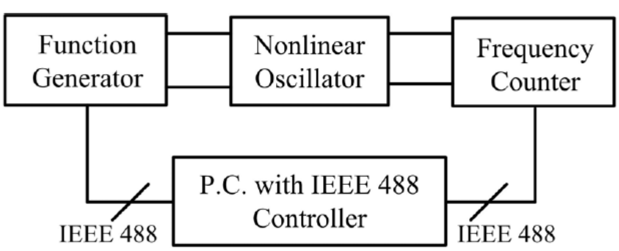
Example: Injection-Locking

140/168

- Experimental measurements



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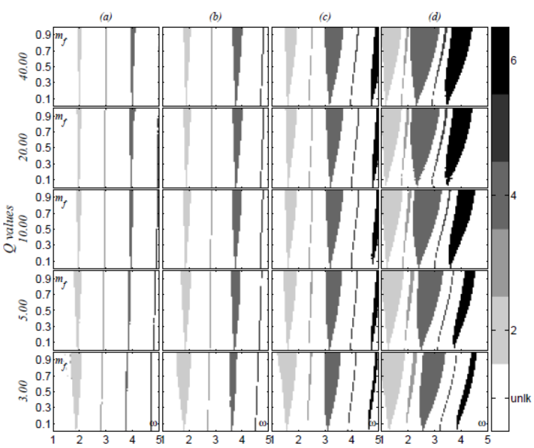
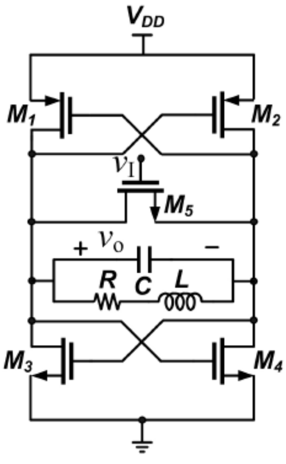
Example: Injection-Locking

- Bifurcation diagram (measured)

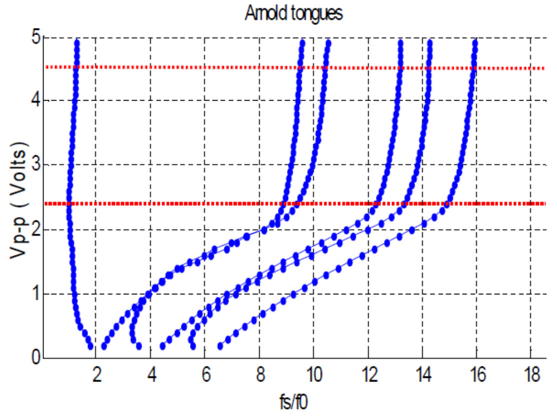


Example: Injection-Locking

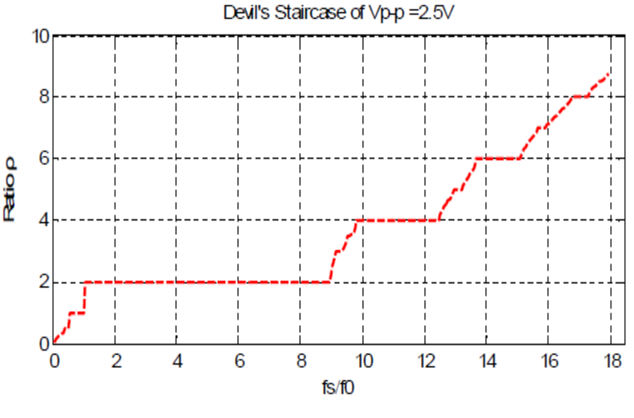
- CMOS ILFD with direct injection



- Bifurcation diagram (measured)



- "Devil's staircase" (measured)



Example: Injection-Locking

- Simulation issues
  - Coexisting solutions
  - Long transients
  - Calculation of frequency ratio
  - Detecting periodicity

Example: Injection-Locking

- Local: Stable limit cycle
- Global: Arnold tongues
- Structural: Fold (?) bifurcation

References: Injection-Locking

[O'DKFQJ05] K. O'Donoghue, M.P. Kennedy, P. Forbes, M. Qu, and S. Jones. A Fast and Simple Implementation of Chua's Oscillator with Cubic-like Nonlinearity. *Int. J. Bif. Chaos*. 15(9):2959-2971, September 2005.

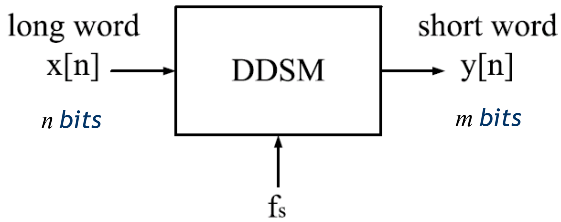
[DdeFK10] S. Daneshgar, O. De Feo and M.P. Kennedy. Observations Concerning the Locking Range in the Complementary Differential LC Injection-Locked Frequency Divider Part I: Qualitative Analysis. *IEEE Trans. Circuits and Systems-Part I*, 57(1):179-188, Jan. 2010.

Example: Digital Delta Sigma Modulator

- Local: unstable equilibrium
- Global: cycles; complex basins of "attraction"
- Structural: multiple nonlinearities

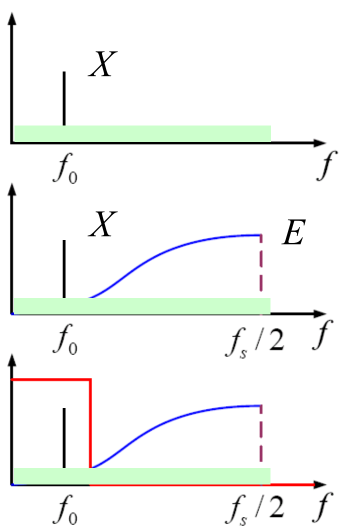
The ideal DDSM

- A bandlimited digital signal x (n bits) is requantized to a shorter word y (m bits)
- The additive quantization noise e is highpass filtered for later removal by a lowpass filter



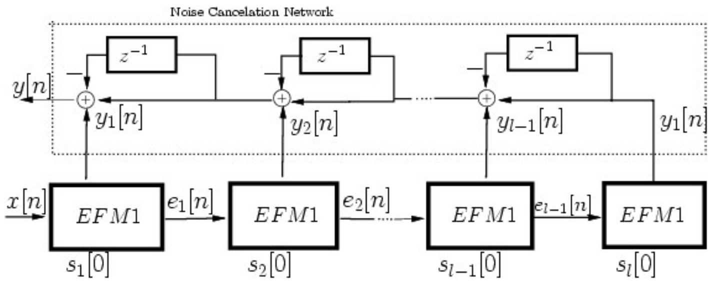
The ideal DDSM

- x is bandlimited
- y includes highpass filtered quantization noise
- quantization noise can be removed by lowpass filtering



- MASH: Higher order filtering with *exact* cancellation of intermediate errors

$$Y(z) = X(z) + E_l(z)\,(1\text{-}z^{-1})^l$$



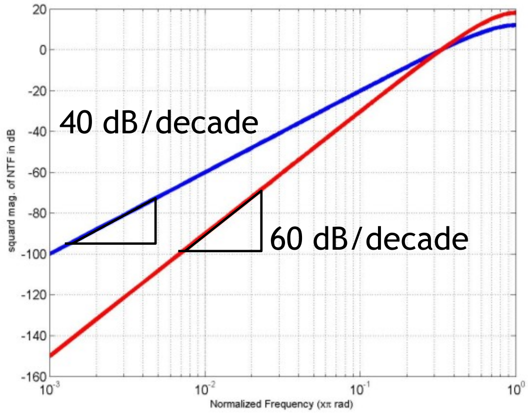
- 3rd order MASH

$$Y_1(z) = X(z) + E_1(z)\,(1\text{-}z^{-1})$$
$$Y_2(z) = \text{-}E_1(z) + E_2(z)\,(1\text{-}z^{-1})$$
$$Y_3(z) = \text{-}E_2(z) + E_3(z)\,(1\text{-}z^{-1})$$

$$Y(z) = Y_1(z) + Y_2(z)\,(1\text{-}z^{-1}) + Y_3(z)\,(1\text{-}z^{-1})^2$$
$$= X(z) + E_3(z)\,(1\text{-}z^{-1})^3$$

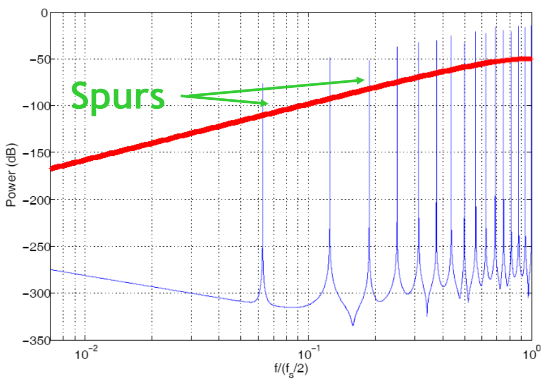
Example: DDSM

- MASH: Idealized power spectra (white noise with 2nd and 3rd order filters)



Example: DDSM

- A MASH 1-1-1 with a constant input can produce *spurious tones* (spurs)



MASH 1-1-1

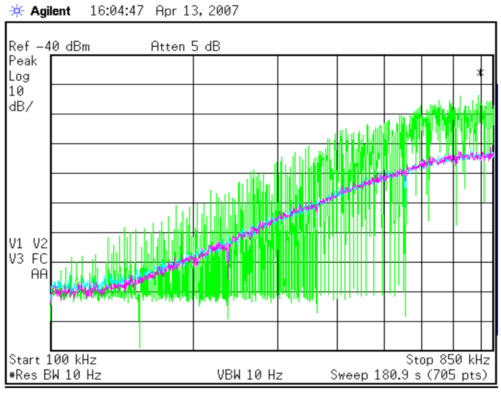
wordlength: 18 bits

Example: DDSM

- Different *wordlengths* can produce different spectra
- Different *initial conditions* can produce different spectra
- Different *inputs* can produce different spectra
- *All of the above* can cause spurs!

Example: DDSM

- Different *wordlengths* produce different spectra



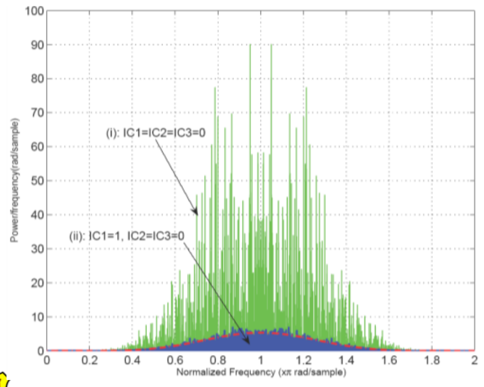
MASH 1-1-1

green plot: 9 bits

blue plot: 18 bits

Example: DDSM

- Different *initial conditions* produce different spectra



MASH 1-1-1

Input = 8 (decimal)

accumulator word length: 14 bits

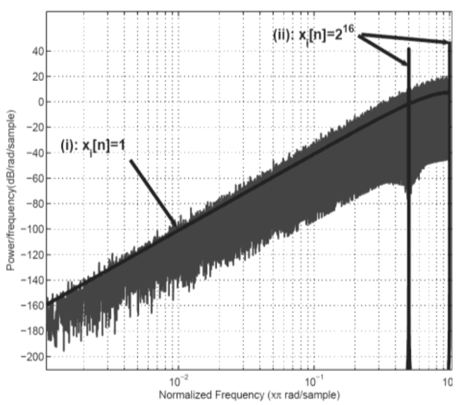
cycle length for green plot: 4096

cycle length for blue plot: 32768

red plot: CMQ approximation

Example: DDSM

- Different *inputs* produce different spectra



MASH 1-1-1

wordlength: 17 bits

(i) $X=1$

(ii) $X=2^{16}$

Example: DDSM

- The DDSM is a Finite State Machine (FSM)
- The FSM has a *finite* state space S (containing N_s states) and a *deterministic* rule G (called the *dynamic*) that governs the evolution of states
- The next state is determined *completely* by the current state and the input:

$$X[n+1] = G(X[n])$$

Example: DDSM

- If the input is fixed, the most complex trajectory visits each state in the state space *once* before repeating; the longest cycle has period $N = N_s\text{-}1$
- In the worst case, the trajectory repeats with period $N = 4$

Stochastic approach

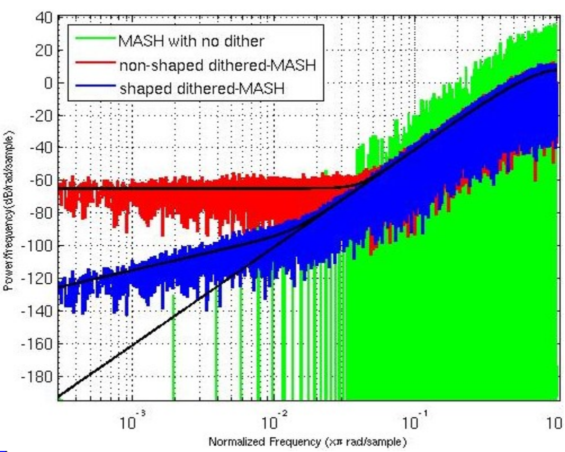
- Make the dynamic *stochastic*
- The next state depends on the current state s, the input x, and a *random* dither signal d

$$s[n+1] = G_S(s[n], x[n], d[n])$$

- Periodicity is destroyed
- Trajectories can be much longer than N_s

Stochastic approach: LSB dither

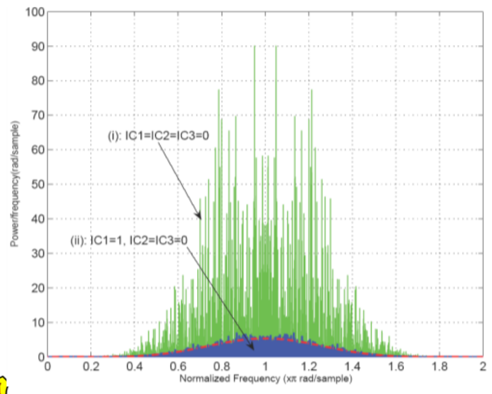
- MASH 1-1-1 with LSB dither: $V(z) = (1-z^{-1})^R$



Green: $V(z) = 0$
Red: $V(z) = 1$
Blue: $V(z) = 1-z^{-1}$

Deterministic approaches: ICs

- MASH 1-1-1: Choose $s_1[0]$ odd (“seeding”)



MASH 1-1-1

Input = 8 (decimal),

accumulator word length: 14 bits

cycle length for green plot: 4096

cycle length for blue plot: 32768

Example: DDSM

- Local: unstable equilibrium
- Global: cycles; complex basins of “attraction”
- Structural: multiple nonlinearities

References: DDSM

[HK07b] K. Hosseini and M.P. Kennedy. Maximum Sequence Length MASH Digital Delta-Sigma Modulators. *IEEE Trans. Circuits and Systems-Part I*, 54(12):2628-2638, Dec. 2007.

[HK08] K. Hosseini and M.P. Kennedy. Architectures for Maximum-Sequence-Length Digital Delta-Sigma Modulators. *IEEE Trans. Circuits and Systems-Part II*, 55(11):1104-1108, Nov. 2008.

[HK11] K. Hosseini and M.P. Kennedy. *Minimizing Tones in Digital Delta-Sigma Modulators.* Springer, New York, 2011.

Summary

- Part 2: Examples
  - SPICE DC analysis (intertwined basins of attraction)
  - Schmitt trigger (saddle-node bifurcation)
  - Colpitts oscillator (Hopf bifurcation)
  - Injection-locking (Arnold tongues)
  - DDSM (coexisting solutions)

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Science Foundation Ireland

Motivation

"I've wondered why it took us so long to catch on.
We saw it and yet we didn't see it.
Or rather *we were trained not to see it*...
The truth knocks on the door and you say, 'Go away, I'm looking for the truth,' and so it goes away..."

Robert M. Pirsig,
Zen and the Art of Motorcycle Maintenance, Corgi, 1974